Relaxation Time in Complex Network

Dmytro Lande
National Technical University
«Igor Sikorsky Kyiv Polytechnic Institute»
Institute for Information Recording of NAS of Ukraine
Kyiv Ukraine
dwlande@email.com

Andrei Snarskii
National Technical University
«Igor Sikorsky Kyiv Polytechnic Institute»
Kyiv Ukraine
asnarskii@gmail.com

Oleh Dmytrenko†
Institute for Information Recording of NAS of Ukraine
Kyiv Ukraine
dmytrenko.o@gmail.com

Igor Subach
National Technical University
«Igor Sikorsky Kyiv Polytechnic Institute»
Kyiv Ukraine
igor_subach@ukr.net

ABSTRACT

New characteristics of nodes of network structures are proposed and researched in this work – the relaxation time of network and the individual relaxation time of node. The so-called decelerated iteration algorithms for HITS and PageRank are used to obtain the relaxation time.

The relaxation time of network and the individual relaxation time of node were tested on the example of a weighted directed network of characters in the novel «Les Miserables». It was found that a topology of a network affects a relaxation of nodes after their perturbation.

As a result of using relaxation time it was possible to identify the most important network components and rank them by the proposed characteristics.

The resulted ranking compared to the ranking by HITS, PageRank and degree of nodes shows the uniqueness of the proposed relaxation time of network and individual relaxation time of node.

CCS CONCEPTS

• Mathematics of computing → Discrete mathematics • Information Systems → Systems and Information Theory • Computing Methodologies → Symbolic and algebraic manipulation → Algorithms

ARES 2020, August 25–28, 2020, Virtual Event, Ireland

ACM ISBN 978-1-4503-8833-7/20/08…$15.00
https://doi.org/10.1145/3407023.3409231

KEYWORDS

Complex network, Perturbation of node scores, Relaxation time of network, Individual relaxation time of node, HITS, PageRank

ACM Reference format:

Dmytro Lande, Andrei Snarskii, Oleh Dmytrenko and Igor Subach. 2020. Relaxation Time in Complex Network. In *The 15th International Conference on Availability, Reliability and Security (ARES 2020), August 25–28, 2020, Virtual Event, Ireland. ACM, New York, NY, USA, 6 pages.* https://doi.org/10.1145/3407023.3409231

1 Introduction

Most of objects in nature and society have a network structure with binary links that can be represented as a network. Topological properties of networks significantly determine their functionality and are the subject of study of complex networks. In many spheres of science and technology, the tasks of analyzing the network topology and research the characteristics of its nodes are important.

The co-called complex network theory deals with the study of the characteristics of complex networks [1, 2]. A powerful mathematical tool of this theory makes it possible to research both the behavior of whole network and separate nodes of such network. Because networking is an effective tool for complex systems researching, network analysis tasks are relevant.

Scientific works by P. Erdősh, M. E. J. Newman, A-L. Barabáshi, S.G. Strogatz, J. M. Kleinberg and other scientists and researchers have made significant contributions to the development of theoretical foundations and practical solutions for the development of methods and tools for the study and design of complex networks.

A number of new methods to solve computationally complex problems specific to modern network structures have been developed [1-3].

Despite the traditional approaches, research of the statistical properties that characterize network behavior; creating network models; predicting the behavior of networks when changing structural properties and with different external influences and perturbations are actual problems of the complex network theory.
In applied research the typical and most important characteristics of network nodes such as the degree of node and characteristics that correspond to the HITS and PageRank algorithms are commonly used for network analysis.
In this paper, in addition to the commonly used network characteristics, new characteristics such as the relaxation time of network and the individual relaxation time of node are proposed and researched.
The aim of this paper is to introduce new characteristics of nodes of networks, determine their "physical sense" and show uniqueness among other characteristics, and provide examples of applications.

2 Iterative HITS and PageRank algorithms

The HITS algorithm (Hyperlink Induced Topic Search) is a ranking algorithm that was proposed and developed by J. M. Kleinberg [4] in 1998. This algorithm calculates the two numbers – an authority and hub weight that determine a document of a certain collection as good authority and hub, respectively.
PageRank algorithm is one of algorithms that calculate the impact factor of web pages and its ranking with using hyperlinks. In 1996 at Stanford University, it was designed and proposed by Sergey Brin and Larry Page [5].
Despite of differences between HITS and PageRank, a common is that the importance (the weight) of node that corresponding to web page or document depends on the weight of other nodes [6, 7]. Also, PageRank, like HITS, is an iterative algorithm based on the linkage of the documents on the web.

3 Relaxation time

In this work new characteristics of nodes of complex network, which make it possible to rank the respective nodes of network, – the relaxation time of network and the individual relaxation time of node are proposed.
The proposed relaxation times are analogous to Maxwell's relaxation time that plays an important role in solid state physics.
The relaxation time τ is the characteristic time during which the electric charge is "absorbed" in an environment with an electrical conductivity σ and a dielectric constant ε. In an infinite homogeneous environment, the heterogeneity of the electric charge distribution is unstable (the system can be gone from an equilibrium state to a non-equilibrium). Over time, the charge is "absorbed", distributed evenly in the environment, and goes away to infinity. The Maxwell's relaxation time τ is the characteristic time of an environment transition into an equilibrium state. The decrease of charge density volume ρ with time t has the form $\rho(t) \sim e^{\frac{-t}{\tau}}$, where $\tau = {}^{\varepsilon}/_{\sigma}$.

By analogy, let introduce the relaxation time of the *k*-th node in the complex system – τ_k. In the beginning, we define the equilibrium state of a complex network as a set of values of nodes s_k^0 ($\overrightarrow{\boldsymbol{s}^0}$ is a vector form) that are determined in accordance with the certain rule, for example in accordance with their scores obtained by the HITS or the PageRank algorithm, or any other [7]. Calculation of $\overrightarrow{\boldsymbol{s}^0}$ in accordance with the selected rule (the iterative algorithm) can always be rewritten in iterative view:

$$\vec{\boldsymbol{s}}(n+1) = \vec{\boldsymbol{s}}(n) + \hat{\boldsymbol{L}}\vec{\boldsymbol{s}}(n),\ n = 0,1,\dots \quad (1)$$

where the node numbers are the vector component numbers of $\vec{\boldsymbol{s}}$; $\hat{\boldsymbol{L}}$ is the operator of the corresponding algorithm (in this work the iterative HITS and PageRank algorithms are considered and used); $\vec{\boldsymbol{s}}(0)$ is the initial scores of nodes.

$$\vec{\boldsymbol{s}}(0) = \overrightarrow{\lim_{n\to\infty} s}(n),\ n = 0,1,\dots \quad (2)$$

and, respectively

$$\hat{\boldsymbol{L}}\vec{\boldsymbol{s}}(0) = 0 \quad (3)$$

Let consider the value of the initial node scores $\vec{\boldsymbol{s}}(0)$ as the solution – $\overrightarrow{\boldsymbol{s}^0}$ (suppose that these scores are equilibrium). Next, we propose to deviate (perturb) the score of the *m*-th node, for example:

$$\vec{\boldsymbol{s}}_i(0) = \vec{\boldsymbol{s}}_i^0 + \alpha\delta_{im}\vec{\boldsymbol{s}}_i^0 \quad (4)$$

where α is the value of the deviation (perturbation) of the *m*-th vector component; δ_{ik} is the Kronecker symbol.
In vector form it means the deviation of from the equilibrium state of one of the components (projections) of the vector $\overrightarrow{\boldsymbol{s}^0}$. The deviation of vector $\overrightarrow{\boldsymbol{s}^0}$ due to deviation of one of its components causes to a non-equilibrium state of system.
Now, the equation (1) for *n*=0 looks as follows:

$$s_i(1) = s_i(0) + \sum_k L_{ik}\, s_i(0), \quad (5)$$

taking into account (4) it can be rewritten as:

$$s_i(1) = s_i(0) + \sum_k L_{ik}\, {s_i}^0 + \alpha \sum_k L_{ik}\, \delta_{km} s_k^0 = {s_i}^0 + \alpha q_i^{(m)}, \quad (6)$$

where the vector $q_i^{(m)} = \sum_k L_{ik}\, \delta_{km} s_k^0$ can be presented as (7) for better visual perception.

$$q_i^{(m)} = \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1m} & \dots & L_{1N} \\ L_{21} & \dots & \dots & L_{2m} & \dots & L_{2N} \\ \vdots & & & \vdots & & \\ L_{m1} & & & \vdots & & \\ \vdots & & & \vdots & & \\ L_{N1} & & & L_{Nm} & & L_{NN} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ s_m^0 \\ \vdots \\ 0 \end{pmatrix} = $$

$$= \begin{pmatrix} L_{1m} \\ L_{2m} \\ \vdots \\ \vdots \\ \vdots \\ L_{1m} \end{pmatrix} s_m^0. \quad (7)$$

For the initial condition $s_i(1)$, $s_i(n)$ converges to the equilibrium state $\overrightarrow{\boldsymbol{s}^0}$ at increasing *n* according to (2).

For the condition $k \neq m$ the initial scores $s_k(0) = s_k^0$. Fig. 1 considers example when after certain step $n \geq 10$ the scores $s_k(n \geq 10)$ becomes smaller than the μ (where μ defines the condition of the convergence).

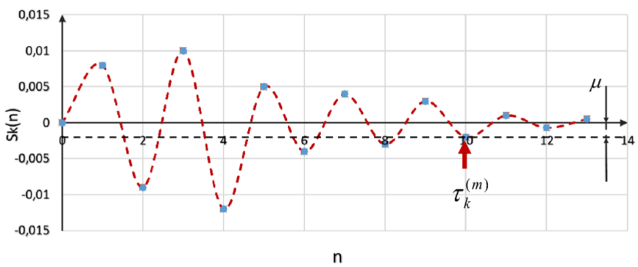

Figure 1: Shema of a gradual convergence of PageRank score of the k-th node after the deviation (perturbation) of the m-th node

The value $\tau_k^{(m)}$ that satisfies the condition (8) for the *k*-th nodes and certain μ will be called the relaxation time of the *k*-th node after the deviation (perturbation) of the *m*-th node.

$$\left| s_k(n \geq \tau_k^{(m)}) \right| \leq \mu. \tag{8}$$

In general, we are interested in the relaxation time of network for the *m*-th node – the highest value of $\tau_k^{(m)}$ for $\forall_k$ after the perturbation of the *m*-th node – $\max_k(\tau_k^{(m)})$. In this work, the individual relaxation time is also proposed and researched. The individual relaxation time is the relaxation time of node that was deviated (perturbed). In this case, we will use the form τ_m avoiding the upper character in $\tau_m^{(m)}$.

4 Research of Relaxation Time

In this work, the proposed characteristics were researched on the example of simple undirected networks and the free-accessible datasets of Social networks [9].

4.1 Example 1

Fig. 2 shows the example of undirected network that consists of 13 nodes and have a structure similar to hierarchical.

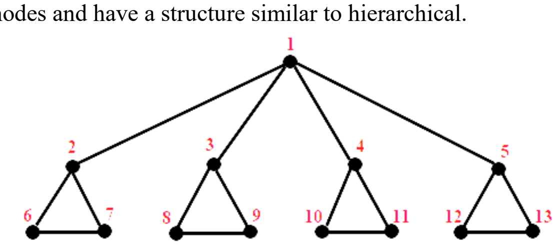

Figure 2: Simple example of undirected network

№	Degree	HITS	Relaxation time of network for HITS	Individual relaxation time of node for HITS
1	8	0.146	25	25
2	6	0.098	26	23
3	6	0.098	26	23
4	6	0.098	26	23
5	6	0.098	26	23
6	4	0.058	21	16
7	4	0.058	21	16
8	4	0.058	21	16
9	4	0.058	21	16
10	4	0.058	21	16
11	4	0.058	21	16
12	4	0.058	21	16
13	4	0.058	21	16

Table 1: The relaxation time of network and the individual relaxation time of node for the network presented in Fig. 2

Considering the individual relaxation time for HITS algorithm, it can be noticed that the node 1 after its perturbation is recovered the longest time. If compare with the relaxation time of the network, we can conclude that the node with number 1 delays the recovery (relaxation) of all system. For example (see fig. 3), node 13 after its perturbation recovers its initial HITS scores at the 16-th step of HITS algorithm, but the node 1 make it at the 21-th step. Detailed analysis of the convergence of each node shows that the node 1 is the most unbalanced and requires the longest time to recover its initial HITS score regardless of which node was perturbed (in other words, for $\forall_k$ $\max_k\left(\tau_k^{(1)}\right) = 25 = \tau_1$, $\max_k\left(\tau_k^{(2)}\right) = 26 = \tau_1$, $\max_k\left(\tau_k^{(3)}\right) = 26 = \tau_1, \dots,$ $\max_k\left(\tau_k^{(13)}\right) = 21 = \tau_1$). In terms of relaxation time of the network, it means that all other nodes wait for the individual relaxation of the node 1. This fact shows that there are some nodes in the network that is the most unbalanced and require the longest time to recover of their initial scores. As a result, all nodes can be ranked by these characteristics.

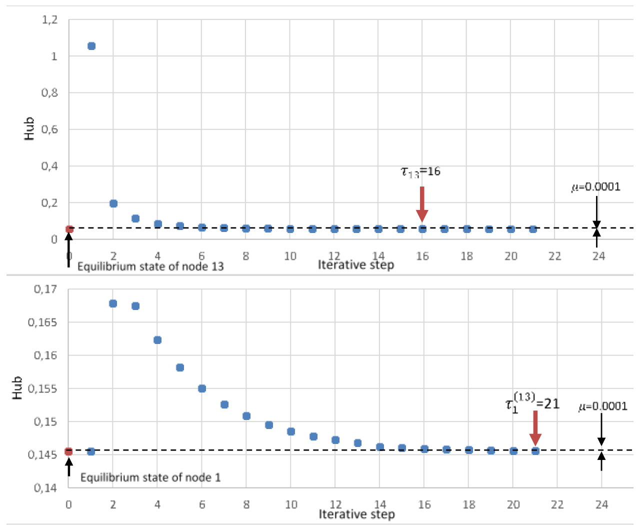

Figure 3: Dynamic of the Hub scores of the nodes 13 and 1 after perturbation the node 13

4.2 Example 2

The relaxation time of network and the individual relaxation time of node are also researched for the weighted network of characters in the novel «Les Miserables» [10].

To research a deceleration factor was equal 0.9 for both iterative HITS and PageRank algorithms (see Annex A). The condition of the convergence was μ = 0.0001 for both HITS and PageRank.

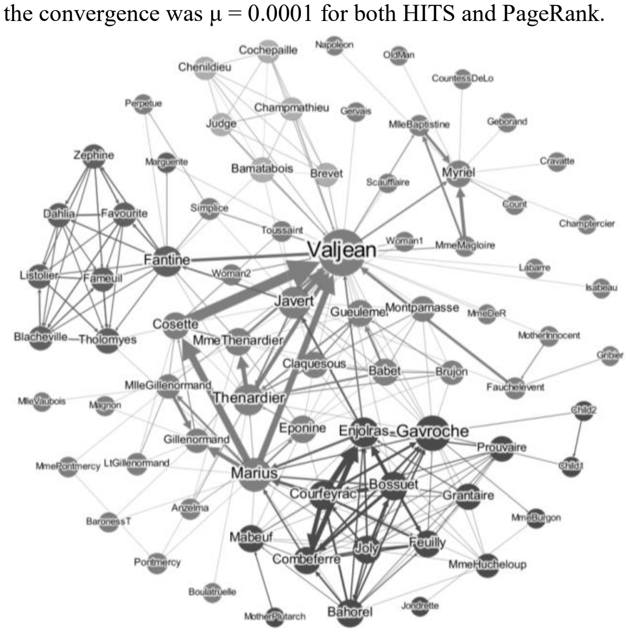

Figure 4: Weighted directed network of characters in the novel «Les Miserables»

Table 2 presents the top 12 of the characters of «Les Miserables» that sorted by decreasing of the relaxation time of network obtained using the decelerated iterative HITS algorithm (see appendix A).

Character	Degree	Hub	Authority	Relaxation time of network for HITS	Individual relaxation time of node for HITS
Courfeyrac	13	0.0126	0.0081	235	215
Bossuet	13	0.0149	0.0015	228	200
Combeferre	11	0.0077	0.0125	221	187
Joly	12	0.0053	0.0002	215	174
Bahorel	12	0.0055	0.0030	212	167
Feuilly	11	0.0036	0.007	206	156
Prouvaire	9	0.0018	0.0033	189	122
Grantaire	10	0.0018	0	189	122
Enjolras	15	0.0277	0.019	187	108
Cosette	11	0.1746	0.1157	170	139
MmeHucheloup	7	0.0009	0	162	93
Javert	17	0.1	0.011	160	116

Table 2: The relaxation time of network and individual relaxation time of node using the decelerated iterative HITS algorithm for the network presented in Fig.4

Also, we present other key characteristics such as degree, hub and authority score for comparison with relaxation time of network and individual relaxation time of node that obtained for weighted directed network of characters in the novel «Les Miserables» using decelerated HITS algorithm. It should be noted that network presented in Fig. 4 is directed and both the hub and authority scores calculated using HITS algorithm have a sense unlike the previous example of undirected network. That is why the authority scores of each node for HITS algorithm are also presented in the table, despite that only the hub score of the *m*-th node was deviated (perturbed).

Also, we can see obvious difference between the relaxation time of network and individual relaxation time of node.

Fig. 5 depicts hub and degree scores of each node and corresponding relaxation time and individual relaxation time calculated using decelerated HITS algorithm for weighted directed network of characters in the novel «Les Miserables». The fig. 5 demonstrates a significant discrepancy between the proposed characteristics and traditional ones. It can be stated that the largest scores of relaxation times are achieved at the average scores of traditional characteristics of the node importance. In other words, this example shows fast recovery to initial scores of HITS of the perturbed nodes that have the low or high degree. But the perturbation of nodes that have the average scores of degree leads to slow recovery to initial scores of HITS.

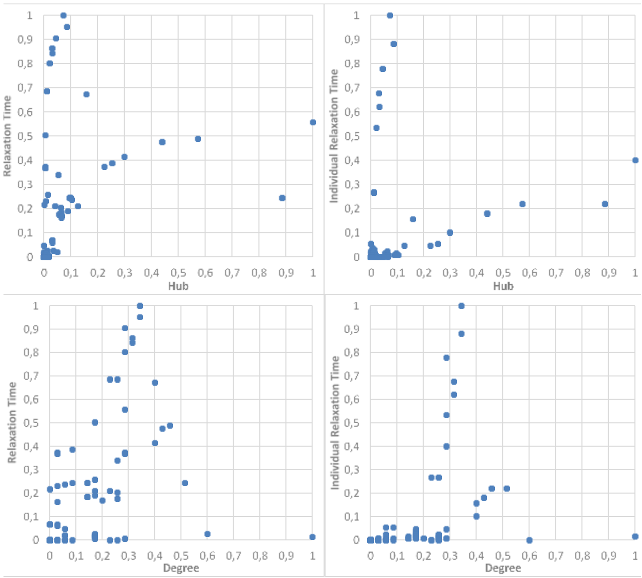

Figure 5: Hub and degree scores of nodes and corresponding relaxation time and individual relaxation time for HITS algorithm (all scores are normalized to [0,1])

Also, the comparison between hub, degree scores and proposed characteristics shows the uniqueness of relaxation time of network and individual relaxation time because the relationship between values of these characteristics is fuzzy.

Next, we consider the relaxation time of network and individual relaxation time calculated using decelerated PageRank algorithm (see appendix A) for the weighted directed network of characters (fig. 4). The results of calculation are presented in the table 3 and sorted by decreasing of the individual relaxation time of nodes.

Character	Degree	PageRank	Relaxation time of network for PageRank	Individual relaxation time of node for PageRank
Myriel	10	0.1319	57	57
Valjean	36	0.1388	57	47
MlleBaptistine	3	0.044	57	34
MmeMagloire	3	0.0335	57	28
Tholomyes	9	0.0327	57	25
Thenardier	16	0.0364	57	24
MmeThenardier	11	0.0300	57	23
Gavroche	22	0.0276	57	19
Cosette	11	0.0234	57	18
Fantine	15	0.0213	57	16

Table 3: The relaxation time of network and individual relaxation time of node calculated using decelerated iterative PageRank algorithm for the network presented in Fig. 4

Fig. 6 depicts PageRank scores of network nodes and corresponding the individual relaxation time calculated using decelerated iterative PageRank algorithm for the network presented in Fig.4.

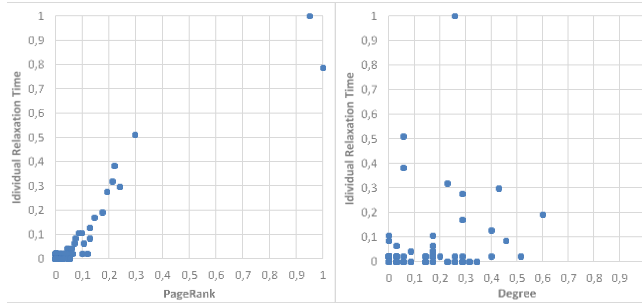

Figure 6: PageRank, degree scores of nodes and corresponded the individual relaxation time for decelerated iterative PageRank algorithm (all scores are normalized to [0,1])

Fig. 6 shows that the individual relaxation time is unique characteristic which is not similar to PageRank or degree.

Analyzing the results presented in the table 3 you can see that the relaxation time for all nodes have the same scores. More detailed analysis shows that node that corresponds to character named Myriel is relaxed the longest time and delays the recovery (relaxation) of all system. In other words, the individual relaxation time of node «Myriel» determines the relaxation time of network despite of which node was perturbed.

To understand how this is happening, it is necessary to consider the dynamic of nodes and research how the PageRank scores of the node «Myriel» changes at each step of decelerated iterative PageRank algorithm. Fig. 7 presents the dynamic of the nodes that corresponds to «Joly» and «Myriel» after perturbation the node that corresponds to «Joly».

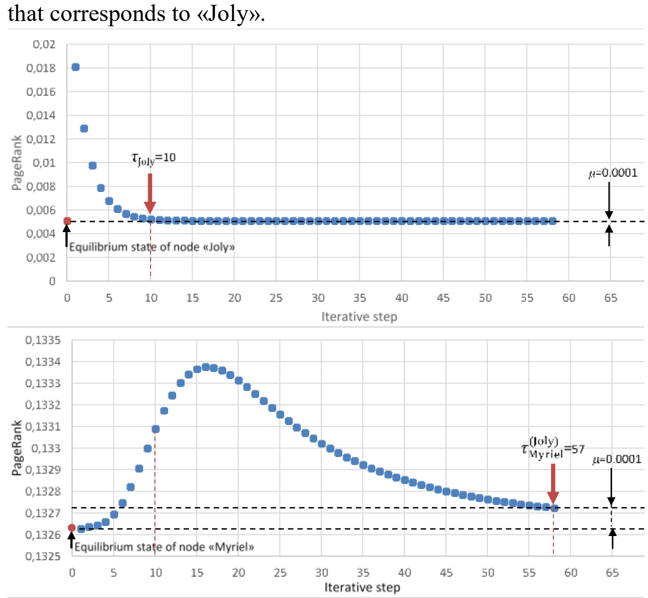

Figure 7: Dynamic of the PageRank scores of the nodes «Joly» and «Myriel» after perturbation the node «Joly»

After perturbation of node that corresponds to «Joly», the node that corresponds to «Myriel» “waits” for relaxation of perturbed node and at that moment increase its PageRank score. But when the node «Joly» came back to its equilibrium state (it happened at the 10-th step of decelerated PageRank algorithm), the PageRank score of the node «Myriel» left its equilibrium state. Now, on the contrary, «Joly» and other nodes of network are “waiting” for relaxation of «Myriel». The relaxation of network after perturbation the node «Joly» occurs only after relaxation of the node «Myriel» at the 57-th iterative step of decelerated PageRank algorithm.

It should be noted that dynamic of each node is unique. Some nodes gradually come back to their initial score recalculated using one of decelerated iterative algorithms (HITS or PageRank). Other can come back to their equilibrium state, but during “waiting” for relaxation of other nodes can again come out of equilibrium and stay in it for a long time. This process is very complex and, all the more, difficult to understand, and in turn depends on a configuration of nodes and, in general, topology of network.

5 Conclusion

In this work, new characteristics of nodes of network structures – the relaxation time of network and the individual relaxation time of node are researched.

Tables 1-3 show that, first, the nodes ranking in accordance with the HITS (PageRank) algorithm compared to the nodes ranking in accordance with the relaxation time calculated using decelerated HITS (PageRank, respectively) algorithm is significantly different. Secondly, the nodes ranking in accordance with the relaxation time that calculated using decelerated HITS (PageRank) algorithm compared to nodes ranking in accordance with degree is also different.

So, the proposed relaxation time of network and individual relaxation time of node are unique characteristics of network. Also, a detailed analysis shows that the relaxation time of network and the individual relaxation time of node characterize resistance of network and, respectively, each of node to perturbations.

It was also found, that the relaxation of some nodes after their perturbation is happening faster than the relaxation of all network.

As a result, the individual relaxation time of node compared to the relaxation time of network is also unique characteristic that characterize separate node after its perturbation.

The condition of the convergence μ is also important during the relaxation time calculating. In this work, the value of μ was chosen so to achieve the better distribution of scores of relaxation time for each node as much as possible.

The relaxation time of network and the individual relaxation time of node were researched on the example of a weighted directed network of characters in the novel «Les Miserables».

The calculation of relaxation time can also be made for other iterative algorithms that are used for computing characteristics of network nodes and researching of network behavior.

So, the proposed characteristics of network nodes can be used to research and analyze complex network. It allows finding and ranking network nodes that recover their equilibrium state after their perturbation for the longer period of time.

ACKNOWLEDGMENTS

This study is funded by the NATO SPS Project CyRADARS (Cyber Rapid Analysis for Defense Awareness of Real-time Situation), Project SPS G5286.

REFERENCES

[1] M.E.J. Newman (2003). The structure and function of complex networks. SIAM Review, 45, 167–256. DOI: http://dx.doi.org/10.1137/S003614450342480.
[2] A.A. Snarskii and D.V. Lande (2015). Modeling of complex networks: tutorial. K.: Engineering.
[3] S.N. Dorogovtsev and J.F. Mendes, 2013 Evolution of networks: from biological networks to the Internet and WWW. Oxford University Press, Oxford, UK.
[4] J.M. Kleinberg (1998). Authoritative sources in a hyperlinked environment. In Processing of ACM-SIAM Symposium on Discrete Algorithms, 46(5), 604–632.
[5] S. Brin and L. Page (1998). The anatomy of a large-scale hypertextual web search engine. Computer networks and ISDN systems, 30(1-7), 107-117
[6] A. Rajaraman and J. D. Ullman, 2011 Mining of massive datasets. Cambridge University Press, Cambridge. DOI: http://dx.doi.org/10.1017/cbo9781139058452.
[7] A.N. Langville and C.D. Meyer, 2011 Google's PageRank and beyond: The science of search engine rankings. Princeton University Press, Princeton.
[8] R. Christensen, 2012 Theory of viscoelasticity: an introduction. Elsevier.
[9] Datasets, https://github.com/gephi/gephi/wiki/Datasets, last accessed 2020/06/14.
[10] D.E. Knuth 1993, The Stanford GraphBase: A Platform for Combinatorial Computing. Addison-Wesley, Reading, MA.

A Deceleration of Iterative Algorithms

The considered above process of calculating the relaxation time is can be successfully applied to reduced matrix. However, when the network has a large number of links, the iterative process of recalculating the scores of the nodes after their perturbation (achieving the equilibrium state) is fast. Enough large number of links connected to the node causes a fast relaxation of the network nodes. As a result, it is enough to make a few iterative steps to achieving the equilibrium state of nodes after perturbation their equilibrium HITS (PageRank) scores. In order to decelerate down the process of convergence to the equilibrium solution of nodes (to their equilibrium HITS or PageRank scores) after their perturbation, we propose to decelerate the algorithms (HITS or PageRank, respectively). After perturbing one of the network nodes, the corresponding iterative HITS or PageRank algorithm with deceleration is applied

$$h_0(i) \to h_1(i) \to h_2(i) \to \ldots \to h_{equilibrium}(i)$$

$$\widehat{\boldsymbol{A}}h_1(i) = h_2(i) \quad \widehat{\boldsymbol{A}}h_n(i) = h_{n+1}(i)$$

$$h_{n+1}(i) \longleftarrow \beta h_{n+1}(i),$$

where $0 < \beta < 1$ is a deceleration factor.

Thus, reducing or increasing the HITS (PageRank) scores of nodes at the appropriate iterative step (deceleration of algorithm), it is possible to achieve the deceleration of network relaxation time.